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Some Doubly Infinite, Finite and Mixed Infinite Sums derived from The N- Fractional Calculus of A Power Function (with Some Examinations)

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Abstract

In this article theorems for some doubly infinite, finite and mixed infinite sums derived from the N-fractional calculus of a power function are reported.

Moreover some numerical examinations for the theorems are reported too.

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D=\{D_{-},D_{+}\}$, $C=\{C_{-},C_{+}\}$,

C_{-} be a curve along the cut joining two points z and $-\infty+i\,\mathrm{Im}(z)$,

C_{+} be a curve along the cut joining two points z and $\infty+i\,\mathrm{Im}(z)$,

D_{-} be a domain surrounded by C_{-} , D_{+} be a domain surrounded by C_{+} .

(Here D contains the points over the curve C).

Moreover, let $f=f(z)$ be a regular function in $D(z\in D)$,

$$f_{\nu}(z)=(f)_{\nu}={}_{C}(f)_{\nu}=\frac{\Gamma(\nu+1)}{2\pi i}\int_{C}\frac{f(\zeta)}{(\zeta-z)^{\nu+1}}d\zeta \qquad (\nu\notin Z^{-}), \tag{1}$$

$$(f)_{-m}=\lim_{\nu\to -m}(f)_{\nu} \qquad (m\in Z^{+}), \tag{2}$$

where $\quad -\pi\leq\arg(\zeta-z)\leq\pi$ for C_{-} , $\quad 0\leq\arg(\zeta-z)\leq 2\pi$ for C_{+} ,

$\zeta\neq z$, $z\in\boldsymbol{C}$, $\nu\in\boldsymbol{R}$, Γ ; Gamma function,

then $(f)_{\nu}$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu>0$, and integrals of order $-\nu$ for $\nu<0$), with respect to z , of the function f , if $|(f)_{\nu}|<\infty$.

(II) On the fractional calculus operator N^{ν} [3]

Theorem A. *Let fractional calculus operator (Nishimoto's Operator) N^{ν} be*

$$N^{\nu} = \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\zeta}{(\zeta - z)^{\nu+1}} \right) \qquad (\nu \notin Z), \quad [\text{Refer to} (1)] \qquad (3)$$

with

$$N^{-m} = \lim_{\nu \to -m} N^{\nu} \qquad (m \in Z^{+}), \qquad (4)$$

and define the binary operation $\circ$ *as*

$$N^{\beta} \circ N^{\alpha} f = N^{\beta} N^{\alpha} f = N^{\beta}(N^{\alpha} f) \qquad (\alpha, \beta \in R), \qquad (5)$$

then the set

$$\left\{ N^{\nu} \right\} = \left\{ N^{\nu} \mid \nu \in R \right\} \qquad (6)$$

is an Abelian product group (having continuous index ν *) which has the inverse transform operator* $(N^{\nu})^{-1} = N^{-\nu}$ *to the fractional calculus operator* N^{ν} *, for the function* f *such that* $f \in F = \left\{ f ;\ 0 \neq |f_{\nu}| < \infty, \nu \in R \right\}$, *where* $f = f(z)$ *and* $z \in C$.
(vis. $-\infty < \nu < \infty$ *).*
(For our convenience, we call $N^{\beta} \circ N^{\alpha}$ as product of N^{β} and N^{α}.)

Theorem B. " F.O.G. $\{N^{\nu}\}$ " *is an* " *Action product group which has continuous index* ν " *for the set of* F . (*F.O.G.* ; Fractional calculus operator group)

Theorem C. *Let*

$$S := \{\pm N^{\nu}\} \cup \{0\} = \{N^{\nu}\} \cup \{-N^{\nu}\} \cup \{0\} \quad (\nu \in R). \qquad (7)$$

Then the set S *is a commutative ring for the function* $f \in F$, *when the identity*

$$N^{\alpha} + N^{\beta} = N^{\gamma} \quad (N^{\alpha}, N^{\beta}, N^{\gamma} \in S) \qquad (8)$$

holds. [5]

(III) Lemma. We have [1]

(i) $$((z-c)^{\beta})_{\alpha} = e^{-i\pi\alpha} \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} (z-c)^{\beta-\alpha} \qquad \left(\left| \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} \right| < \infty \right),$$

(ii) $$(\log(z-c))_{\alpha} = -e^{-i\pi\alpha} \Gamma(\alpha)(z-c)^{-\alpha} \qquad (|\Gamma(\alpha)| < \infty),$$

(iii) $$((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z-c) \qquad (|\Gamma(\alpha)| < \infty),$$

where $z - c \neq 0$ in (i), and $z - c \neq 0, 1$ in (ii) and (iii). (Γ; Gamma function),

(iv) $$(u \cdot v)_{\alpha} = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha+1-k)} u_{\alpha-k} v_k \qquad \left(\begin{array}{l} u = u(z), \\ v = v(z) \end{array} \right).$$

§ 1. Doubly Infinite, Finite and Mixed Sums

In the following $\alpha, \beta, \gamma \in \boldsymbol{R}$.

Theorem 1. *Let*

$$G(\alpha,\beta,\gamma\ ;k,m) := \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(m-\beta)\Gamma(k-m-\alpha+\gamma)}{k!\cdot m!\,\Gamma(\alpha+1-k)\Gamma(\gamma+1-m)\Gamma(-\beta)\Gamma(k-\alpha)}. \tag{1}$$

(i) *When* $\alpha, \beta, \gamma \notin \boldsymbol{Z}$, *we have the following doubly infinite sums ;*

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} G(\alpha,\beta,\gamma\ ;k,m)\left(\frac{z-c}{z}\right)^m\left(\frac{c}{z-c}\right)^k \subseteq \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\left(\frac{z-c}{z}\right)^{\gamma-\alpha}, \tag{2}$$

where

$$|(z-c)/z|<1, \quad |c/(z-c)|<1,$$

and

$$\left|\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\right|, \quad \left|\frac{\Gamma(k-\alpha+\gamma-m)}{\Gamma(k-\alpha)}\right| < \infty \ .$$

The identity (notation $=$ *) holds for* $(\gamma-\alpha) \in \boldsymbol{Z}$.

(ii) *When* $\alpha, \gamma \notin \boldsymbol{Z}^+$, *we have the following mixed infinite sums ;*

$$\sum_{k=0}^{\infty}\sum_{m=0}^{s} G(\alpha,s,\gamma\ ;k,m)\left(\frac{z-c}{z}\right)^m\left(\frac{c}{z-c}\right)^k \subseteq \frac{\Gamma(\gamma-\alpha-s)}{\Gamma(-\alpha-s)}\left(\frac{z-c}{z}\right)^{\gamma-\alpha}, \tag{3}$$

for $s \in \boldsymbol{Z}^+$ *where*

$$|c/(z-c)|<1, \quad |(z-c)/z|<\infty,$$

and

$$\left|\frac{\Gamma(\gamma-\alpha-s)}{\Gamma(-\alpha-s)}\right|, \quad \left|\frac{\Gamma(k-\alpha+\gamma-m)}{\Gamma(k-\alpha)}\right| < \infty \ .$$

The identity (notation $=$ *) holds for* $(\gamma-\alpha) \in \boldsymbol{Z}$.

Proof of (i). We have

$$z^\alpha = (z-c)^\alpha\left(1-\frac{c}{c-z}\right)^\alpha \tag{4}$$

$$= (z-c)^\alpha \sum_{k=0}^{\infty} \frac{(-c)^k\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)}(c-z)^{-k} \quad \left(|c-z|>|-c|\right) \tag{5}$$

$$= \sum_{k=0}^{\infty} \frac{c^k \Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)} (z-c)^{\alpha-k} \quad . \tag{6}$$

Next make (6) $\times z^{\beta}$, then operate N- fractional calculus operator N^{γ} to its both sides, we obtain

$$(z^{\alpha+\beta})_{\gamma} = \sum_{k=0}^{\infty} \frac{c^k \Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)} \left((z-c)^{\alpha-k} \cdot z^{\beta}\right)_{\gamma} \tag{7}$$

$$= \sum_{k=0}^{\infty} \frac{c^k \Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)} \sum_{m=0}^{\infty} \frac{\Gamma(\gamma+1)}{m!\,\Gamma(\gamma+1-m)} \left((z-c)^{\alpha-k}\right)_{\gamma-m} (z^{\beta})_m , \tag{8}$$

by Lemma (i i).

Now we have

$$(z^{\alpha+\beta})_{\gamma} = e^{-i\pi\gamma} \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} z^{\alpha+\beta-\gamma} \qquad \left(\left| \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \right| < \infty \right) , \tag{9}$$

$$\left((z-c)^{\alpha-k}\right)_{\gamma-m} = e^{-i\pi(\gamma-m)} \frac{\Gamma(k-\alpha+\gamma-m)}{\Gamma(k-\alpha)} (z-c)^{\alpha-k-\gamma+m} , \tag{10}$$

$$\left(\left| \frac{\Gamma(k-\alpha+\gamma-m)}{\Gamma(k-\alpha)} \right| < \infty \right)$$

and

$$(z^{\beta})_m = e^{-i\pi m} \frac{\Gamma(m-\beta)}{\Gamma(-\beta)} z^{\beta-m} \tag{11}$$

by Lemma (i), respectively.

Therefore, substituting (9), (10) and (11) into (8) we obtain

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} G(\alpha,\beta,\gamma\,;k,m) \left(\frac{z-c}{z}\right)^{m} \left(\frac{c}{z-c}\right)^{k} = \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \left(\frac{z-c}{z}\right)^{\gamma-\alpha} . \tag{12}$$

However the LHS (left hand side) of (12) is always one valued function, on the contrary the RHS (right hand side) of (12) is many valued function for $(\gamma-\alpha) \notin \boldsymbol{Z}$ and one valued one for $(\gamma-\alpha) \in \boldsymbol{Z}$.

Hence we must calculate as

$$\left(\frac{z-c}{z}\right)^{\gamma-\alpha} = \left(e^{i2n\pi} \frac{z-c}{z}\right)^{\gamma-\alpha} \qquad \begin{pmatrix} (\gamma-\alpha) \notin \boldsymbol{Z} \\ n \in \boldsymbol{Z} \end{pmatrix} , \tag{13}$$

because we are now being in the field of complex analysis.

Moreover when $(\gamma-\alpha)\in\boldsymbol{Z}$ both of the LHS and the RHS of (12) are one valued functions respectively. In this case we have (12) strictly.

Therefore, we have (2) from (12), considering (13) for $(\gamma-\alpha)\notin\boldsymbol{Z}$.

Proof of (i i). Set $\beta = s\in\boldsymbol{Z}^+$ in (2), we have then (3) clearly, under the conditions.

Corollary 1. *When* $r, p\in\boldsymbol{Z}^+$ *we have the doubly finite sums;*

$$\sum_{k=0}^{r}\sum_{m=0}^{p} G(r,\beta,p\,;k,m)\left(\frac{z-c}{z}\right)^m\left(\frac{c}{z-c}\right)^k = \frac{\Gamma(p-r-\beta)}{\Gamma(-r-\beta)}\left(\frac{z-c}{z}\right)^{p-r}, \quad (14)$$

for $(p-r)\in\boldsymbol{Z}$, *where*

$$|c/(z-c)|\,,\quad |(z-c)/z|<\infty,$$

and

$$\left|\frac{\Gamma(p-r-\beta)}{\Gamma(-r-\beta)}\right|,\quad \left|\frac{\Gamma(k-r+p-m)}{\Gamma(k-r)}\right|<\infty\ .$$

Proof . Set $\alpha=r$ and $\gamma=p$ in (2), we have then this corollary clearly.

Now both of the LHS and RHS of (14) are one valued functions respectively, hence the identity (notation $=$) holds in (2).

Corollary 2. *When* $r, p\in\boldsymbol{Z}^+$ *we have the doubly finite sums;*

$$\sum_{k=0}^{r}\sum_{m=0}^{s} G(r,s,p\,;k,m)\left(\frac{z-c}{z}\right)^m\left(\frac{c}{z-c}\right)^k = \frac{\Gamma(p-r-s)}{\Gamma(-r-s)}\left(\frac{z-c}{z}\right)^{p-r}, \quad (15)$$

where

$$|c/(z-c)|\,,\quad |(z-c)/z|<\infty,$$

and

$$\left|\frac{\Gamma(p-r-s)}{\Gamma(-r-s)}\right|,\quad \left|\frac{\Gamma(k-r+p-m)}{\Gamma(k-r)}\right|<\infty\ .$$

Proof . Set $\alpha=r$ and $\gamma=p$ in (3), we have then this corollary under the conditins, clearly.

Now both of the LHS and RHS of (15) are one valued functions respectively, hence the identity (notation $=$) holds in (3).

§ 2. Direct calculation of the doubly infinite sums

The direct calculation (without the use of N- fractional calculas) of the doubly infinite sum in LHS of § 1. (2) is shown as follows.

Theorem 2. *Let*

$$G = G(\alpha,\beta,\gamma\ ;k,\ m) := \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(m-\beta)\Gamma(k-m-\alpha+\gamma)}{k!\cdot m!\,\Gamma(\alpha+1-k)\Gamma(\gamma+1-m)\Gamma(-\beta)\Gamma(k-\alpha)}\ , \qquad (1)$$

and

$$P = P(\alpha,\beta,\gamma) := \frac{\sin\pi\alpha\cdot\sin\pi(\gamma-\alpha-\beta)}{\sin\pi(\alpha+\beta)\cdot\sin\pi(\gamma-\alpha)}\ . \qquad (2)$$

We have then

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} G\cdot\left(\frac{z-c}{z}\right)^m\left(\frac{c}{z-c}\right)^k = P\cdot\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\left(\frac{z-c}{z}\right)^{\gamma-\alpha}\ , \qquad (3)$$

where

$$|\,c/(z-c)\,| < 1,$$

and

$$(\alpha+\beta),\ (\gamma-\alpha),\ (\gamma-\alpha-\beta) \notin \mathbf{Z}\ .$$

Proof. Now we have

$$G\cdot\left(\frac{z-c}{z}\right)^m\left(\frac{c}{z-c}\right)^k$$

$$= \frac{\Gamma(\gamma-\alpha)}{\Gamma(-\alpha)}\cdot\frac{[-\beta]_m[-\gamma]_m[\gamma-\alpha-m]_k}{k!\cdot m!\,[1+\alpha-\gamma]_m}\left(\frac{z-c}{z}\right)^m\left(\frac{c}{z-c}\right)^k \qquad (4)$$

using the identity

$$\Gamma(\lambda+1-k) = (-1)^{-k}\frac{\Gamma(\lambda+1)\Gamma(-\lambda)}{\Gamma(k-\lambda)} \qquad (5)$$

and

$$\Gamma(k+\gamma-\alpha-m) = (-1)^{-m}\Gamma(\gamma-\alpha)\frac{[\gamma-\alpha-m]_k}{[1+\alpha-\gamma]_m}\ , \qquad (6)$$

where

$$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda)\text{, with } [\lambda]_0 = 1$$

(notation of Pochhammer).

We have then

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} G\cdot\left(\frac{z-c}{z}\right)^m\left(\frac{c}{z-c}\right)^k = \frac{\Gamma(\gamma-\alpha)}{\Gamma(-\alpha)}$$

$$\times\sum_{m=0}^{\infty}\frac{[-\beta]_m[-\gamma]_m}{m!\,[1+\alpha-\gamma]_m}\left(\frac{z-c}{z}\right)^m\sum_{k=0}^{\infty}\frac{[\gamma-\alpha-m]_k}{k!}\left(\frac{-c}{z-c}\right)^k \qquad (7)$$

$$=\frac{\Gamma(\gamma-\alpha)}{\Gamma(-\alpha)}\left(\frac{z}{z-c}\right)^{\alpha-\gamma}\sum_{m=0}^{\infty}\frac{[-\beta]_m[-\gamma]_m}{m!\,[1+\alpha-\gamma]_m} \tag{8}$$

$$=\frac{\Gamma(\gamma-\alpha)}{\Gamma(-\alpha)}\left(\frac{z}{z-c}\right)^{\alpha-\gamma}\ {}_2F_1(-\beta,-\gamma\,;1+\alpha-\gamma\,;1) \tag{9}$$

$$=\frac{\Gamma(\gamma-\alpha)\Gamma(1+\alpha-\gamma)\Gamma(1+\alpha+\beta)}{\Gamma(-\alpha)\Gamma(1+\alpha)\Gamma(1+\alpha+\beta-\gamma)}\left(\frac{z-c}{z}\right)^{\gamma-\alpha}, \tag{10}$$

where

$$\left|\frac{c}{z-c}\right|<1,\ \left|\frac{z-c}{z}\right|<1,\ \operatorname{Re}(\alpha+\beta)>-1\ .$$

Because we ahave

$$\sum_{k=0}^{\infty}\frac{[\gamma-\alpha-m]_k}{k!}\left(\frac{-c}{z-c}\right)^k=\left(\frac{z}{z-c}\right)^{\alpha-\gamma}\left(\frac{z}{z-c}\right)^m \tag{11}$$

since

$$\sum_{k=0}^{\infty}\frac{[\lambda]_k}{k!}z^k=(1-z)^{-\lambda}, \tag{12}$$

and

$${}_2F_1(a,b;c\,;1)=\sum_{m=0}^{\infty}\frac{[a]_m[b]_m}{m!\,[c]_m}=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\quad\begin{pmatrix}\operatorname{Re}(c-a-b)>0\\ c\notin \mathbf{Z}_0^-\end{pmatrix}. \tag{13}$$

Moreover we have the identity.

$$\Gamma(\lambda)\Gamma(1-\lambda)=\frac{\pi}{\sin\pi\lambda}\qquad(\lambda\notin\mathbf{Z}), \tag{14}$$

then applying (14) to (10) we obtain (3) under the conditions.

§ 3. Some Numerical Examinations for Theorem 1

[I] Examination of Theorem 1. (2) (Doubly infinite sum)

Set

$$c=1,\quad z=3,\quad \alpha=1/4\quad\text{and}\quad \beta=\gamma=1/2$$

in Theorem 1. (2) we obtain

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}G(1/4,1/2,1/2\,;k,m)\left(\frac{1}{2}\right)^k\left(\frac{2}{3}\right)^m\subseteq\frac{\Gamma(-1/4)}{\Gamma(-3/4)}\left(\frac{2}{3}\right)^{1/4} \tag{1}$$

$$=3\cdot\frac{\Gamma(3/4)}{\Gamma(1/4)}\left(e^{i2n\pi}\frac{2}{3}\right)^{1/4} \qquad (n\in Z) \qquad (2)$$

$$=\begin{cases} 0.91622\cdots & (\text{for } n=0) \quad (3) \\ i\cdot 0.91622\cdots & (\text{for } n=1) \quad (4) \\ -0.91622\cdots & (\text{for } n=2) \quad (5) \\ -i\cdot 0.91622\cdots & (\text{for } n=3) \quad (6) \end{cases}$$

Now the LHS of (1) is real, then we must choose (3) and (5) from the set { (3), (4), (5), (6) }.

And now we have

$$G(1/4,1/2,1/2\,;\,0\,,0\,)\left(\frac{1}{2}\right)^0\left(\frac{2}{3}\right)^0 \qquad \left(\begin{array}{c}\text{first term of}\\ \text{the LHS of (1)}\end{array}\right)$$

$$=\frac{\Gamma(1/4)}{\Gamma(-1/4)}=-\frac{1}{4}\cdot\frac{\Gamma(1/4)}{\Gamma(3/4)}<0\ . \qquad (7)$$

Then choosimg (5) from the set { (3), (5) } , since the sign of the double infinite sum of the LHS of (1) is decided by the sign of its first term (with $k=m=0$), when

$$\left|G_{k,m}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^m\right|>\left|G_{k+1,\,m+1}\left(\frac{1}{3}\right)^{k+1}\left(\frac{2}{3}\right)^{m+1}\right|, \quad G_{k,m}=G,$$

we have then

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} G(1/4,1/2,1/2\,;k,m)\left(\frac{1}{2}\right)^k\left(\frac{2}{3}\right)^m=-0.91622\cdots \qquad (5)$$

from (1) , considering (7).

Indeed we have

$$\text{LHS of (5)}=\sum_{k=0}^{\infty}\frac{\Gamma(\frac{5}{4})}{k\,!\,\Gamma(\frac{5}{4}-k)}\left(\frac{1}{2}\right)^k\left\{\frac{\Gamma(k+\frac{1}{4})}{\Gamma(k-\frac{1}{4})}\right.$$

$$+\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})\Gamma(k-\frac{3}{4})}{\Gamma(\frac{1}{2})\Gamma(-\frac{1}{2})\Gamma(k-\frac{1}{4})}\left(\frac{2}{3}\right)+\frac{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})\Gamma(k-\frac{7}{4})}{2!\,\Gamma(-\frac{1}{2})\Gamma(-\frac{1}{2})\Gamma(k-\frac{1}{4})}\left(\frac{2}{3}\right)^2$$

$$+\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})\Gamma(k-\frac{11}{4})}{3!\,\Gamma(-\frac{3}{2})\Gamma(-\frac{1}{2})\Gamma(k-\frac{1}{4})}\left(\frac{2}{3}\right)^{3}+\frac{\Gamma(\frac{3}{2})\Gamma(\frac{7}{2})\Gamma(k-\frac{15}{4})}{4!\,\Gamma(-\frac{5}{2})\Gamma(-\frac{1}{2})\Gamma(k-\frac{1}{4})}\left(\frac{2}{3}\right)^{4}$$

$$+\frac{\Gamma(\frac{3}{2})\Gamma(\frac{9}{2})\Gamma(k-\frac{19}{4})}{5!\,\Gamma(-\frac{7}{2})\Gamma(-\frac{1}{2})\Gamma(k-\frac{1}{4})}\left(\frac{2}{3}\right)^{5}+\frac{\Gamma(\frac{3}{2})\Gamma(\frac{11}{2})\Gamma(k-\frac{23}{4})}{6!\,\Gamma(-\frac{9}{2})\Gamma(-\frac{1}{2})\Gamma(k-\frac{1}{4})}\left(\frac{2}{3}\right)^{6}+\cdots\cdots\Bigg\} \tag{8}$$

$$=\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\left\{-\frac{1}{4}+\frac{1}{2\cdot4^{2}}-\frac{5}{2!\,2^{2}\cdot4^{3}}+\frac{5\cdot9}{3!\,2^{3}\cdot4^{4}}-\frac{5\cdot9\cdot13}{4!\,2^{4}\cdot4^{5}}+\frac{5\cdot9\cdot13\cdot17}{5!\,2^{5}\cdot4^{6}}-\cdots\right\}$$

$$+\frac{\Gamma(\frac{1}{4})}{3\,\Gamma(\frac{3}{4})}\left\{-\frac{1}{2\cdot3}-\frac{1}{2^{2}\cdot4}+\frac{1}{2!\,2^{3}\cdot4^{2}}-\frac{5}{3!\,2^{4}\cdot4^{3}}+\frac{5\cdot9}{4!\,2^{5}\cdot4^{4}}-\frac{5\cdot9\cdot13}{5!\,2^{6}\cdot4^{5}}-\cdots\right\}$$

$$+\frac{\Gamma(\frac{1}{4})}{2!\,2^{2}\cdot3^{2}\,\Gamma(\frac{3}{4})}\left\{-\frac{4}{3\cdot7}-\frac{1}{2\cdot3}-\frac{1}{2!\,2^{2}\cdot4}+\frac{1}{3!\,2^{3}\cdot4^{2}}-\frac{5}{4!\,2^{4}\cdot4^{3}}+\frac{5\cdot9}{5!\,2^{5}\cdot4^{4}}-\cdots\right\}$$

$$-\frac{\Gamma(\frac{1}{4})}{3!\,2^{3}\cdot3\,\Gamma(\frac{3}{4})}\left\{\frac{4^{2}}{3\cdot7\cdot11}+\frac{4}{2\cdot3\cdot7}+\frac{1}{2!3}+\frac{1}{3!\,2^{3}\cdot4}\right.$$
$$\left.-\frac{1}{4!\,2^{4}\cdot4^{2}}+\frac{5}{5!\,2^{5}\cdot4^{3}}-\frac{5\cdot9}{6!\,2^{6}\cdot4^{4}}+\cdots\right\}$$

$$+\frac{5^{2}\,\Gamma(\frac{1}{4})}{4!\,2^{4}\cdot3^{2}\,\Gamma(\frac{3}{4})}\left\{-\frac{4^{3}}{3\cdot7\cdot11\cdot15}-\frac{4^{2}}{2\cdot3\cdot7\cdot11}-\frac{4}{2!\,2^{2}\cdot3\cdot7}-\frac{1}{3!\,2^{3}\cdot3}\right.$$
$$\left.-\frac{1}{4!\,2^{4}\cdot4}+\frac{1}{5!\,2^{5}\cdot4^{2}}-\cdots\right\}$$

$$-\frac{5^{2}\cdot7^{2}\,\Gamma(\frac{1}{4})}{5!\,2^{5}\cdot3^{3}\,\Gamma(\frac{3}{4})}\left\{\frac{4^{4}}{3\cdot7\cdot11\cdot15\cdot19}+\frac{4^{3}}{2\cdot3\cdot7\cdot11\cdot15}+\frac{4^{2}}{2!\,2^{2}\cdot3\cdot7\cdot11}\right.$$
$$\left.+\frac{4}{3!\,2^{3}\cdot3\cdot7}+\frac{1}{4!\,2^{4}\cdot3}+\frac{1}{5!\,2^{5}\cdot4}+\cdots\cdots\right\}+\cdots\cdots \tag{9}$$

$$=-(0.66861\cdots)-(0.22272\cdots)-(0.01591\cdots)-(0.00690\cdots)$$
$$-(0.00180\cdots)-(0.00093\cdots)-\cdots\cdots \tag{10}$$

$$=-0.9168\cdots\quad. \tag{11}$$

[II] Examination of Theorem 1. (3) (Mixed infinite sum)

Set

$$c=1, \quad z=3, \quad \alpha=1/4, \quad \gamma=1/2 \quad \text{and} \quad s=1$$

in Theorem 1. (3) we obtain

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} G(1/4,1,1/2\,;k,m)\left(\frac{1}{2}\right)^k\left(\frac{2}{3}\right)^m \subseteq \frac{\Gamma(-3/4)}{\Gamma(-5/4)}\left(\frac{2}{3}\right)^{1/4} \qquad (12)$$

$$= -\frac{5}{12}\cdot\frac{\Gamma(1/4)}{\Gamma(3/4)}\left(e^{i2n\pi}\frac{2}{3}\right)^{1/4} \qquad (n\in\mathbf{Z}) \qquad (13)$$

$$=\begin{cases} -1.113943\cdots & (\text{for } n=0) \qquad (14)\\ -i\cdot 1.113943\cdots & (\text{for } n=1) \qquad (15)\\ 1.113943\cdots & (\text{for } n=2) \qquad (16)\\ i\cdot 1.113943\cdots & (\text{for } n=3) \qquad (17).\end{cases}$$

Now the LHS of (1) is real, then we must choose (14) and (16) from the set { (14), (15), (16), (17) }.

And now we have

$$G(1/4,1,1/2\,;0,0)\left(\frac{1}{2}\right)^0\left(\frac{2}{3}\right)^0 \qquad \left(\begin{matrix}\text{first term of}\\ \text{the LHS of (12)}\end{matrix}\right)$$

$$=\frac{\Gamma(1/4)}{\Gamma(-1/4)} = -\frac{1}{4}\cdot\frac{\Gamma(1/4)}{\Gamma(3/4)} < 0 \ . \qquad (18)$$

Then choosimg (14) from the set { (14), (16) }, since the sign of the double infinite sum of the LHS of (12) is decided by the sign of its first term (with $k=m=0$), when

$$\left|G_{k,m}\left(\frac{1}{2}\right)^k\left(\frac{2}{3}\right)^m\right| > \left|G_{k+1,m+1}\left(\frac{1}{2}\right)^{k+1}\left(\frac{2}{3}\right)^{m+1}\right| , \quad G_{k,m}=G(\alpha,s,\gamma\,;k,m),$$

we have then

$$\sum_{k=0}^{\infty}\sum_{m=0}^{1} G(1/4,1,1/2\,;k,m)\left(\frac{1}{2}\right)^k\left(\frac{2}{3}\right)^m = -1.113943\cdots , \qquad (19)$$

from (13), considering (18).

Indeed we have

$$\text{LHS of (12)} = \sum_{k=0}^{\infty} \frac{\Gamma(\frac{5}{4})}{k!\,\Gamma(\frac{5}{4}-k)\Gamma(k-\frac{1}{4})}\left(\frac{1}{2}\right)^{k}$$

$$\times \sum_{m=0}^{1} \frac{\Gamma(\frac{3}{2})\Gamma(m-1)\Gamma(k-m+\frac{1}{4})}{m!\,\Gamma(\frac{3}{2}-m)\Gamma(-1)}\left(\frac{2}{3}\right)^{m} \tag{20}$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(\frac{5}{4})\Gamma(k+\frac{1}{4})}{k!\,\Gamma(\frac{5}{4}-k)\Gamma(k-\frac{1}{4})}\left(\frac{1}{2}\right)^{k}$$

$$+ \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})\Gamma(\frac{5}{4})\Gamma(k-\frac{3}{4})}{k!\,\Gamma(\frac{5}{4}-k)\Gamma(k-\frac{1}{4})}\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)^{k} \tag{21}$$

$$= \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\left\{-\frac{1}{4}+\frac{1}{2\cdot 4^2}-\frac{5}{2!\,2^2\cdot 4^3}+\frac{5\cdot 9}{3!\,2^3\cdot 4^4}-\frac{5\cdot 9\cdot 13}{4!2^4\cdot 4^5}+\frac{5\cdot 9\cdot 13\cdot 17}{5!\,2^5\cdot 4^6}-\cdots\right\}$$

$$-\frac{\Gamma(\frac{1}{4})}{3\,\Gamma(\frac{3}{4})}\left\{\frac{1}{3}+\frac{1}{8}-\frac{1}{2!8^2}+\frac{5}{3!\,8^3}-\frac{5\cdot 9}{4!\,8^4}+\frac{5\cdot 9\cdot 13}{5!\,8^5}-\cdots\right\} \tag{22}$$

$$= \frac{\Gamma(1/4)}{\Gamma(3/4)}\{-0.25+0.03125-(0.009765\cdots)+(0.003662\cdots)$$

$$-(0.00487\cdots)+(0.000632\cdots)-\cdots\}$$

$$-\frac{1}{3}\cdot\frac{\Gamma(1/4)}{\Gamma(3/4)}\{(0.333333\cdots)+0.125-(0.007812\cdots)+(0.001627\cdots)$$

$$-(0.000457\cdots)+(0.000148\cdots)-\cdots\} \tag{23}$$

$$= (-0.667796\cdots)+(-0.445583\cdots) \tag{24}$$

$$= -1.113379\cdots \quad . \tag{25}$$

[I I I] Examination of Corollary 1 (Doubly finite sum)

Set

$$c=1,\quad z=3,\quad r=3,\quad \beta=-1/2 \quad \text{and} \quad p=2$$

in Corollary 1, we obtain

$$\sum_{k=0}^{3}\sum_{m=0}^{2} G(3,-1/2,2\,;k,m)\left(\frac{1}{2}\right)^{k}\left(\frac{2}{3}\right)^{m} = \frac{\Gamma(-1/2)}{\Gamma(-5/2)}\left(\frac{2}{3}\right)^{-1} = \frac{45}{8} \tag{26}$$

Indeed we have

$$\text{LHS of (26)} = \sum_{k=0}^{3} \frac{3!\cdot 2!}{k!\,\Gamma(4-k)} \left(\frac{1}{2}\right)^{k} \sum_{m=0}^{2} \frac{\Gamma(m+\frac{1}{2})\Gamma(k-m-1)}{m!\,\Gamma(3-m)\Gamma(\frac{1}{2})\Gamma(k-3)} \left(\frac{2}{3}\right)^{m} \tag{27}$$

$$= \sum_{k=0}^{3} \frac{6\Gamma(k-1)}{k!\,\Gamma(4-k)\Gamma(k-3)} \left(\frac{1}{2}\right)^{k} + \sum_{k=0}^{3} \frac{4\Gamma(k-2)}{k!\,\Gamma(4-k)\Gamma(k-3)} \left(\frac{1}{2}\right)^{k}$$

$$+ \sum_{k=0}^{3} \frac{2}{k!\,\Gamma(4-k)} \left(\frac{1}{2}\right)^{k} \tag{28}$$

$$= \left\{ \frac{\Gamma(-1)}{\Gamma(-3)} + \frac{3}{2} \cdot \frac{\Gamma(0)}{\Gamma(-2)} + \frac{3}{4} \cdot \frac{1}{\Gamma(-1)} + \frac{1}{8} \cdot \frac{1}{\Gamma(0)} \right\}$$

$$+ \left\{ \frac{2}{3} \cdot \frac{\Gamma(-2)}{\Gamma(-3)} + \frac{\Gamma(-1)}{\Gamma(-2)} + \frac{1}{2} \cdot \frac{\Gamma(0)}{\Gamma(-1)} + \frac{1}{12} \cdot \frac{1}{\Gamma(0)} \right\}$$

$$+ \left\{ \frac{1}{3} + \frac{1}{2} + \frac{1}{4} + \frac{1}{24} \right\} \tag{29}$$

$$= \{6+3+0+0\} + \{-2-2-(1/2)+0\} + (9/8) = \frac{45}{8} \ . \tag{30}$$

In this example both of the LHS and RHS of (26) are one valued functions respectively, then we have the identity (notation $=$), without the ad hoc shown in [I] and [I I].

§ 4. Commentary

[I] Applying N- fractional calculus to some power functions we obtain the result shown by **§ 1.** (2).

On the other hand we obtain the identity **§ 2.** (3), by the direct calculation of its LHS. However

the LHS of **§ 2.** (3) is always one valued function,

on the contrary

the RHS of **§ 2.** (3) is many valued function for $(\gamma-\alpha) \notin \mathbf{Z}$,

and one valued one for $(\gamma-\alpha) \in \mathbf{Z}$.

We have then

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} G \cdot \left(\frac{z-c}{z}\right)^{m} \left(\frac{c}{z-c}\right)^{k} \subseteq P \cdot \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \left(\frac{z-c}{z}\right)^{\gamma-\alpha} \tag{1}$$

from **§ 2.** (3), considering **§ 1.** (13), in complex analysis.

[I I] Therefore, when

$$|P| = |P(\alpha, \beta, \gamma)| = 1, \tag{2}$$

§ 1. (2) and (1) overlaps each other, because

$$\left\{ \textit{Set of the elements of } \left(\frac{z-c}{z}\right)^{\gamma-\alpha} \right\} = \left\{ \textit{Set of the elements of } -\left(\frac{z-c}{z}\right)^{\gamma-\alpha} \right\} \tag{3}$$

$$((\gamma-\alpha) \notin \mathbf{Z}).$$

That is, the result **§ 1.** (2) which is obtained by the use of N- fractional calculus is a special case of (1) which is the result obtained by the direct calculation of its LHS.

Namely the space $|P| = |P(\alpha, \beta, \gamma)| = 1$ **in which the result § 1. (2) holds is a subspace of the space such that** $|P| = |P(\alpha, \beta, \gamma)| = M < \infty$ **in which the result (1) holds good.**

Note, When

$$c = 1, \quad z = 3, \quad \gamma = 1/2, \quad \alpha = 1/4,$$

we have

$$\text{LHS of (3)} = \left\{ (2/3)^{1/4},\ i(2/3)^{1/4},\ -(2/3)^{1/4},\ -i(2/3)^{1/4} \right\}$$

and

$$\text{RHS of (3)} = \left\{ -(2/3)^{1/4},\ -i(2/3)^{1/4},\ (2/3)^{1/4},\ i(2/3)^{1/4} \right\},$$

for example.

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